

Geometrical Aspects of non-gravitational interactions

Omar Roldan^{1,2}, C.C. Barros Jr.¹

¹Universidade Federal de Santa Catarina, ²Universidade Federal do Rio de Janeiro

Abstract

In this work we look for a geometric description of non-gravitational forces. The basic ideas are proposed studying the interaction between a punctual particle and an electromagnetic external field. For this purpose, we introduce the concept of proper space-time, that allow us to describe this interaction in a way analogous to the one that the general relativity theory does for gravitation. The field equations that define this geometry are similar to the Einstein's equations, where in general, the energy-momentum tensor have information of both, the particle and the external field. In this formalism we consider the particle path as being a geodesic in a curved space-time, and so, the electromagnetic force is understood in a purely geometric way.

Introduction

As our knowledge of Physics improves, more and more the general theory of relativity is showing to be a fundamental theory. Among many important results that could be listed, the recent detection of gravitational waves by the LIGO experiment [1] is just one example.

Although, when considering non-gravitational interactions there are still some questions to be answered. An interesting aspect to be observed is the preferential role of the gravitational interactions in this theory, as all interactions are coupled to the gravity by the gravitational constant G in the right-hand side of the Einstein equations. A fundamental question is if the gravitational interaction really is a preferential kind of interaction or if all interactions may be considered in an equivalent way. Another question that may be posed is how a system where the gravity is negligible if compared with the other kinds of interactions may be considered in a geometrical approach. Some interesting results of these ideas at the quantum level have been shown in [2] and [3]. In this work, this possibility will be studied and the formulation of the basic ideas will be made in terms of the interaction of a point-like particle with an external electromagnetic field.

The theory of General Relativity (GR) describes the gravitation as a geometrical effect. In this way, in absence of non-gravitational forces¹, any particle will follow a geodesic path on a given curved space-time (hereafter s-t) as described by the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0. \quad (1)$$

Being a geometrical effect, it implies that the evolution of each particle in a gravitational field is independent on their characteristics such as charge, mass or spin.

¹In this work, our main interest is in studying interactions other than gravitation.

A different situation is the case of a charged particle interacting with an electromagnetic field (hereafter EM field), in which the trajectory is determined by

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = \frac{1}{m} f^\mu, \quad (2)$$

where m is the mass of the particle and f^μ is the Lorentz force which depends on the particle's charge. Then, when introducing interactions other than gravitation in the usual formulation of GR, the dynamics of particles is not (completely) described in geometrical terms.

In this work, we want to *effectively* describe the electromagnetic interaction as a geometrical phenomena. That is, we begin with the assumption that the path followed by a charged particle in presence of an EM field, correspond to a geodesic in a certain curved s-t, which we will call the particle's proper space-time (proper s-t). Thinking in this way, each particle is in free fall in its own s-t, which must necessarily depend on the interaction between the particle and its environment (the external fields).

Then, we must carefully distinguish between this abstract s-t (the proper s-t) and the s-t which is perceived by the observer in the laboratory (our s-t). We will call this last one the background space-time (background s-t).

To our knowledge, there is no theory at present which totally describes the electromagnetic interaction in a geometric way. Many people have care about this issue, moved for example by Einstein's spirit which in its own words is: "The idea that there are two structures of space independent of each other, the metric-gravitational and the electromagnetic, is intolerable to the theoretical spirit" (Aspden [4], and Barnett [5]).

Hermann Weyl (Pauli [6], see also [7]), developed a generalized version of Riemann geometry to describe the electromagnetism in a geometrical way. Even though it was a very deep theory in which he has able to obtain the Maxwell equations by just using geometry, this formalism was practically abandoned, as it predicts some effects which are not consistent with observations (Pauli [6])

It's not the purpose of this work to give a geometric description of electromagnetism, but just to gain some insights on how to proceed in order to effectively describe the interaction of a charged particle and an EM field. Some useful results can be obtained with this approach as it was done for example, in the work of Barros [2, 3, 8], in which by using the spherical symmetry in the interaction of electron with a proton in the Hydrogen atom, he managed to obtain a Minkowski-like metric and it allows him to obtain the spectrum of the system which is very close to the observed one.

This work is organized in four sections. In section 1. we make the weak field approximation and the limit of small velocity to get a first relationship between the metric of the proper s-t, the external EM field and the particle mass and charge. Section 2. sets the main ideas on how to obtain the metric for the proper s-t in more general cases. Section 3 and 4. give the applications of the present formalism and its comparison with the results of the Special Relativity (SR). In the last section the conclusions will be drawn.

1 Weak field and small velocity limit

1.1 Geodesic equation and Lorentz force

The first step in order to construct the proper s-t is to determine it in the weak field limit and its relation with a flat s-t, in a way similar to the one that is done in the General Relativity Theory.

In the weak field approximation we can think of the proper s-t as described by a metric $g_{\mu\nu}$ which is a small perturbation of the background s-t. For simplicity we shall take this background s-t to be flat and so described by the Minkowski metric² $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \quad (3)$$

Here $h_{\mu\nu}$ is understood as a two-tensor in the background s-t, and is responsible for the force on the charged particle (as perceived by the observer). The inverse metric at linear order in h is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad h^{\mu\lambda} = \eta^{\mu\nu} \eta^{\lambda\sigma} h_{\nu\sigma}, \quad (4)$$

that is, we rise and lower indices with the background metric.

The Christoffel symbols $\Gamma_{\rho\sigma}^\mu$ for the metric g are expressed into the usual form

$$\Gamma_{\rho\sigma}^\mu = \frac{1}{2} g^{\mu\lambda} (-g_{\rho\sigma,\lambda} + g_{\lambda\rho,\sigma} + g_{\sigma\lambda,\rho}), \quad (5)$$

which can be related to the Christoffel symbols $\tilde{\Gamma}_{\rho\sigma}^\mu$ of the background metric η by³

$$\begin{aligned} \Gamma_{\rho\sigma}^\mu &= \frac{1}{2} (\eta^{\mu\lambda} - h^{\mu\lambda}) \left[-(\eta_{\rho\sigma,\lambda} + h_{\rho\sigma,\lambda}) + (\eta_{\lambda\rho,\sigma} + h_{\lambda\rho,\sigma}) + (\eta_{\sigma\lambda,\rho} + h_{\sigma\lambda,\rho}) \right] \\ &= \tilde{\Gamma}_{\rho\sigma}^\mu - A_{\rho\sigma}^\mu, \end{aligned} \quad (6)$$

where we introduced

$$A_{\rho\sigma}^\mu \equiv \frac{1}{2} h^{\mu\lambda} [\eta_{\rho\sigma,\lambda} + \eta_{\lambda\rho,\sigma} + \eta_{\sigma\lambda,\rho}] - \frac{1}{2} \eta^{\mu\lambda} [-h_{\rho\sigma,\lambda} + h_{\lambda\rho,\sigma} + h_{\sigma\lambda,\rho}]. \quad (7)$$

Unlike $\tilde{\Gamma}_{\rho\sigma}^\mu$, $A_{\rho\sigma}^\mu$ is a tensor, as we now see. To prove this, it is easier to consider $A_{\mu\rho\sigma}$ instead of $A_{\rho\sigma}^\mu$,

$$\begin{aligned} A_{\mu\rho\sigma} &= -\frac{1}{2} (-h_{\rho\sigma,\mu} + h_{\mu\rho,\sigma} + h_{\sigma\mu,\rho}) + \tilde{\Gamma}_{\rho\sigma}^\lambda h_{\mu\lambda} \\ &= -\frac{1}{2} (-\nabla_\mu h_{\rho\sigma} + \nabla_\sigma h_{\mu\rho} + \nabla_\rho h_{\sigma\mu}), \end{aligned} \quad (8)$$

where ∇ means the covariant derivative with respect to the background s-t, that is

$$\nabla_\mu h_{\rho\sigma} = \partial_\mu h_{\rho\sigma} - \tilde{\Gamma}_{\mu\rho}^\lambda h_{\lambda\sigma} - \tilde{\Gamma}_{\mu\sigma}^\lambda h_{\rho\lambda}, \quad (9)$$

and the statement about $A_{\mu\rho\sigma}$ is verified. Now the main idea of the proper s-t concept, is that the particle will follow a geodesic in this curved s-t, so the particle's path should obey

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0, \quad (10)$$

or

$$\frac{d^2 x^\mu}{d\tau^2} + \tilde{\Gamma}_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = A_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}. \quad (11)$$

²This approach is common in perturbation theory for the study of gravitational waves, see: [9, 10, 11, 12].

³In Cartesian coordinates $\tilde{\Gamma}_{\rho\sigma}^\mu$ vanishes, but this is not the case in spherical coordinates for example. Using $\tilde{\Gamma}_{\rho\sigma}^\mu$ explicitly allows us to easily generalize our present treatment to a curved background s-t.

If we take the limit of small velocities the proper time τ in the background s-t and in the proper s-t are the same, and we can set $\tau = t$, where t is the time as measured by the observer. In addition⁴, $dx^i/d\tau \ll dx^0/d\tau \approx c$, then by comparing Eq. (11) with Eq. (2) we can say that the observer in the laboratory will measure a force f^μ given by⁵

$$f^i = mA^i_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = m \left[A^i_{00} \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} + 2A^i_{0j} \frac{dx^0}{d\tau} \frac{dx^j}{d\tau} \right], \quad (12)$$

where we used the fact that $A^\mu_{\rho\sigma}$ is symmetric in $\rho\sigma$.

To continue, let's stress that we are considering an inertial observer in a flat background s-t, whose metric in Cartesian coordinates is

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1). \quad (13)$$

Now, if this observer choose any other coordinate system to describe the spatial part, say, spherical coordinates, but without changing the temporal part, then

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \eta_{11} & \eta_{12} & \eta_{13} \\ 0 & \eta_{21} & \eta_{22} & \eta_{23} \\ 0 & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix}, \quad (14)$$

where in general $\eta_{ij} = \eta_{ij}(\vec{x}) \neq 0$ for $i \neq j$. Note that if we make a boost this will affect the time component, but as far as we take the limit of small velocities, we are somehow performing a Galilean transformation of coordinates, which does not affect the time component of the metric.

Considering this convention we have that in general $\tilde{\Gamma}^\mu_{\rho\sigma} \neq 0$, but

$$\tilde{\Gamma}^0_{\mu\sigma} = \tilde{\Gamma}^\lambda_{\mu 0} = \tilde{\Gamma}^\lambda_{0\sigma} = 0. \quad (15)$$

Using this fact we can easily compute

$$\begin{aligned} \nabla_0 h_{\rho\sigma} &= \partial_0 h_{\rho\sigma}, \quad \nabla_i h_{00} = \partial_i h_{00}, \\ \nabla_i h_{0j} &= \partial_i h_{0j} - \tilde{\Gamma}^k_{ij} h_{0k}, \end{aligned} \quad (16)$$

so that

$$A_{i00} = \partial_i \frac{h_{00}}{2} - \partial_0 h_{i0}, \quad A_{ij0} = \frac{1}{2} (\partial_i h_{j0} - \partial_j h_{i0}) - \frac{1}{2} \partial_0 h_{ij}. \quad (17)$$

If we relate $h_{\mu 0}$ with the electromagnetic potential through

$$h_{00} = 2 \frac{q}{mc} A_0, \quad h_{0i} = \frac{q}{mc} A_i, \quad (18)$$

we see that Eq. (17) yields

$$A_{i00} = \frac{q}{mc} F_{i0}, \quad A_{ij0} = \frac{q}{2mc} F_{ij} - \frac{1}{2} \partial_0 h_{ij}, \quad (19)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the EM tensor. Substituting Eq. (19) into (12) give us

$$f^i = q [cF^i_0 + F^i_j v^j] - q \partial_0 h^i_j v^j, \quad (20)$$

⁴We are using $x^0 = ct$.

⁵Note that we are only considering the spatial component i of the force, as the temporal component would require to consider $t \neq \tau$.

which is the Lorentz force $f^i = qF^i_\rho dx^\rho/d\tau$ plus an additional term, $q \partial_0 h^i_j$, which we expect to be subdominant due to the time derivative $\partial_0 = \frac{1}{c}\partial_t$ (at least for slowly time-varying EM fields) and vanishes for the cases of interest in the present paper: static EM fields. The more general situation will be addressed in a future paper.

We want to stress that the EM potential A_μ is a vector in the background s-t, and by construction $h_{\mu\nu}$ is a tensor. Then, it seems that the quantity $(h_{00}/2, h_{0i})$ cannot be identified with a vector as we suggested in Eq. (18). However, as it was said, in this paper we only allow spatial coordinates transformations (or Galilean transformations)⁶

$$x^\mu = (x^0, x^k) \rightarrow x^{\mu'} = (x^0, x^{k'}), \quad (21)$$

and so, A_μ transform as

$$A_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} A_\mu = \left(A_0, \frac{\partial x^{k'}}{\partial x^k} A_k \right). \quad (22)$$

The transformation for $h_{0\mu}$ is given by

$$\begin{aligned} h_{0\mu'} &= \frac{\partial x^\nu}{\partial x^0} \frac{\partial x^\mu}{\partial x^{\mu'}} h_{\nu\mu} = \frac{\partial x^\mu}{\partial x^{\mu'}} h_{0\mu} \\ &\rightarrow h_{0\mu'} = \left(h_{00}, \frac{\partial x^k}{\partial x^{k'}} h_{0k} \right), \end{aligned} \quad (23)$$

that is, $h_{0\mu}$ has the same behavior as A_μ under this restricted group of transformations⁷. We leave the discussion about a possible generalization to any group of transformations for a future paper.

1.2 Riemann Tensor and gauge freedom

The Riemann tensor for the proper s-t is

$$R^\mu{}_{\nu\rho\sigma} = -\Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\sigma\nu,\rho} - \Gamma^\lambda_{\nu\rho} \Gamma^\mu_{\sigma\lambda} + \Gamma^\lambda_{\sigma\nu} \Gamma^\mu_{\rho\lambda}, \quad (24)$$

which can be split as

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma} &= -\left(\tilde{\Gamma}^\mu_{\nu\rho,\sigma} - A^\mu{}_{\nu\rho,\sigma} \right) + \left(\tilde{\Gamma}^\mu_{\sigma\nu,\rho} - A^\mu{}_{\sigma\nu,\rho} \right) \\ &\quad - \left(\tilde{\Gamma}^\lambda_{\nu\rho} - A^\lambda{}_{\nu\rho} \right) \left(\tilde{\Gamma}^\mu_{\sigma\lambda} - A^\mu{}_{\sigma\lambda} \right) + \left(\tilde{\Gamma}^\lambda_{\sigma\nu} - A^\lambda{}_{\sigma\nu} \right) \left(\tilde{\Gamma}^\mu_{\rho\lambda} - A^\mu{}_{\rho\lambda} \right). \end{aligned}$$

In first order in h (and in its derivatives) we can write $R^\mu{}_{\nu\rho\sigma} = (R_0 + R_1)^\mu{}_{\nu\rho\sigma}$, where R_0 refers to the Riemann tensor in the background s-t, so for our case $(R_0)^\mu{}_{\nu\rho\sigma} = 0$ (flat s-t). Then,

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma} &= (R_1)^\mu{}_{\nu\rho\sigma} = A^\mu{}_{\nu\rho,\sigma} - A^\mu{}_{\sigma\nu,\rho} + \tilde{\Gamma}^\lambda_{\nu\rho} A^\mu{}_{\sigma\lambda} \\ &\quad + A^\lambda{}_{\nu\rho} \tilde{\Gamma}^\mu_{\sigma\lambda} - \tilde{\Gamma}^\lambda_{\sigma\nu} A^\mu{}_{\rho\lambda} - A^\lambda{}_{\sigma\nu} \tilde{\Gamma}^\mu_{\rho\lambda}, \end{aligned} \quad (25)$$

which can be written in a compact form as

$$R^\mu{}_{\nu\rho\sigma} = \nabla_\sigma A^\mu{}_{\nu\rho} - \nabla_\rho A^\mu{}_{\sigma\nu}. \quad (26)$$

⁶E.g., we can go from Cartesian (t, x, y, z) to spherical coordinates (t, r, θ, ϕ) .

⁷We can also note that this property applies to the covariant derivative. That is, from Eq. (16) we see that h_{00} behaves like a scalar while h_{0i} transform like a vector.

From Eq. (11) we see that the quantity we can directly measure is $A^\mu{}_{\rho\sigma}$, once this is directly related to the force f^μ or to the particle's path. Then, in accordance with Eq. (8) we can think of $h_{\mu\sigma}$ as an interacting potential. This leads necessarily to a gauge freedom for the tensor $h_{\mu\sigma}$ and so for the metric $g_{\mu\sigma}$.

We can then ask, which is the more general “group” of transformations on $h_{\mu\nu}$ that leaves the dynamics of the particle invariant as perceived by the observer. To start with, we can seek for which transformations leave the Riemann tensor invariant. This was already solved (see e.g., Carroll [9])

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \quad (27)$$

where ξ_μ is an arbitrary vector⁸ in the background s-t. However, $A^\mu{}_{\rho\sigma}$ is not invariant under this general transformation as we will see now. From Eq. (8) we have

$$A_{\mu\rho\sigma} \rightarrow \frac{1}{2} \left\{ -\nabla_\mu (h_{\rho\sigma} + \nabla_\rho \xi_\sigma + \nabla_\sigma \xi_\rho) + \nabla_\sigma (h_{\mu\rho} + \nabla_\mu \xi_\rho + \nabla_\rho \xi_\mu) + \nabla_\rho (h_{\sigma\mu} + \nabla_\sigma \xi_\mu + \nabla_\mu \xi_\sigma) \right\},$$

which leads to

$$A_{\mu\rho\sigma} \rightarrow A_{\mu\rho\sigma} - \nabla_\rho \nabla_\sigma \xi_\mu, \quad (28)$$

and we have used $\nabla_\rho \nabla_\sigma = \nabla_\sigma \nabla_\rho$ which is valid for a flat s-t, as we assumed to be the case for the background s-t.

Using this transformation we can easily see that the Riemann tensor is invariant. Substituting into Eq. (26) we get

$$\begin{aligned} R_{\mu\nu\rho\sigma} &\rightarrow \nabla_\sigma (A_{\mu\nu\rho} - \nabla_\nu \nabla_\rho \xi_\mu) - \nabla_\rho (A_{\mu\sigma\nu} - \nabla_\sigma \nabla_\nu \xi_\mu) \\ &= R_{\mu\nu\rho\sigma}. \end{aligned} \quad (29)$$

Though $A_{\mu\rho\sigma}$ is not invariant under the full group of transformations (Eq. (27)), if we fix the three spatial components ξ^i , then the gauge freedom is due only to ξ^0 and we get the restricted transformation

$$h_{00} \rightarrow h_{00} + 2\nabla_0 \xi_0, \quad h_{0i} \rightarrow h_{0i} + \nabla_i \xi_0, \quad h_{ij} \rightarrow h_{ij}. \quad (30)$$

Defining $\xi_0 = (q/mc) \phi$, we have the gauge transformation

$$\left(\frac{h_{00}}{2}, h_{0i} \right) \rightarrow \left(\frac{h_{00}}{2} + \nabla_0 \phi, h_{0i} + \nabla_i \phi \right), \quad (31)$$

which is perfectly consistent with the gauge transformation $A_\mu \rightarrow A_\mu + \nabla_\mu \phi$, see Eq. (18).

Under this restricted group of transformations the tensor A transform according to (see Eq. (28))

$$A_{0\rho\sigma} \rightarrow A_{0\rho\sigma} - \nabla_\rho \nabla_\sigma \xi_0, \quad A_{i\rho\sigma} \rightarrow A_{i\rho\sigma}, \quad (32)$$

and since $A_{i\rho\sigma}$ is invariant, then the force f_i (see Eq. (12))

$$f_i = mA_{i\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}, \quad (33)$$

will also be invariant. However the temporal component

$$f^0 = c \frac{dp^0}{d\tau} = mc A^0{}_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}, \quad (34)$$

⁸The only restriction on ξ is that is small enough such that the condition $|h_{\mu\nu}| \ll 1$ is satisfied.

which represents the energy transfer is not invariant. We leave the solution of this problem for a future paper.

In section 1, we used the limit of weak field and small velocities to obtain a relationship between the proper s-t metric and the external EM field. In the next section we go an step further and to try to tackle the more general case in which the fields and velocities do not need be small.

2 Electromagnetic field in the proper s-t

In Special Relativity and in Cartesian coordinates, the electromagnetic tensor $F^{\mu\nu}$ is totally defined by the electric and magnetic fields \vec{E} , \vec{B} (See [9, 13, 14, 15])

$$F^{0i} \equiv E^i/c, \quad F_{ij} \equiv \tilde{\epsilon}_{ijk} B^k, \quad (35)$$

where $\tilde{\epsilon}_{ijk}$ is the totally antisymmetric Levi-Civita symbol, with $\tilde{\epsilon}_{123} = 1$. However, in the general case, the tensor $F^{\mu\nu}$ is not completely defined by \vec{E} and \vec{B} , since we need to know the metric (in this case, $\eta_{\mu\nu}$) in order to know F_{0i} and F^{ij} . Let's study this in more detail.

In a curved s-t we need to use the Levi-Civita tensor, which in four dimensions is given by (see e.g Carroll [9])

$$\epsilon_{\mu\nu\rho\sigma} \equiv \sqrt{-g} \tilde{\epsilon}_{\mu\nu\rho\sigma}, \quad g = \det(g_{\mu\nu}), \quad (36)$$

and once more $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ is totally antisymmetric with $\tilde{\epsilon}_{0123} = 1$. The Levi-Civita tensor with upper indexes is

$$\epsilon^{\mu\nu\rho\sigma} \equiv \frac{\tilde{\epsilon}^{\mu\nu\rho\sigma}}{\sqrt{-g}}, \quad \tilde{\epsilon}^{0123} = -1. \quad (37)$$

Then, the electric field determine the F^{0i} components, while magnetic field determine (except by the factor $\sqrt{-g}$) the other components F_{ij} . So, in general

$$F^{0i} \equiv E^i/c, \quad F_{ij} \equiv \epsilon_{0ijk} B^k. \quad (38)$$

Nonetheless, a complete knowledge of the EM tensor is only possible with the help of the metric, e.g., to know $F_{0i} = g_{0\mu} g_{i\nu} F^{\mu\nu}$ we need not only the metric g as well as the components F^{ij} , which also depend on $F_{\mu\nu}$ by $F^{ij} = g^{i\mu} g^{j\nu} F_{\mu\nu}$.

Then, we see that in general the EM tensor $F_{\mu\nu}$ will depend on the particle's properties (because of the dependence on $g_{\mu\nu}$) but since the electric and magnetic fields \vec{E} e \vec{B} are external fields

$$E^i = cF^{0i}, \quad \text{and} \quad B^i \equiv \frac{1}{2} \epsilon^{0ijk} F_{jk}, \quad (39)$$

they are independent on the particle's charge and mass. \vec{E} and \vec{B} are not unknown quantities, they are imposed as external conditions.

To proceed, we shall give three essential statements for a general proper s-t:

The first one is that for vanishing external EM field ($\vec{E} = \vec{B} = 0$) the proper s-t must coincide with the background s-t.

The second one is that, from the point of view of the proper s-t formalism, the charged particle follows a geodesic in a curved s-t which is determined by the interaction with the external EM fields \vec{E} and \vec{B} , so it is natural that the EM tensor built through Eq. (39) must satisfy the Maxwell equations in the proper s-t. That is,

$$\nabla_\mu F^{\nu\mu} = \mu_0 J^\nu, \quad (40)$$

where hereafter ∇_μ is the covariant derivative in the proper s-t, and J^ν is the current density which sources the EM field. In addition, in the present work we are interested in the case in which the charged particle is moving in the vacuum (not in a medium) so we can set $J^\nu = 0$. Note that in the background s-t J^ν need not be zero everywhere, but it is only non-zero in places where our particle of interest is not moving on.

And as a third statement, we will consider that if we apply a force f^i to this particle⁹ the equation of motion should be

$$\frac{d^2 x^i}{d\tau^2} + \Gamma_{\rho\sigma}^i \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = \frac{f^i}{m}. \quad (41)$$

Now, if we take f^i such that the particle stays at rest, then

$$0 = \frac{d^2 x^i}{d\tau^2} = -\Gamma_{00}^i \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} + \frac{f^i}{m}, \quad (42)$$

or

$$0 = \frac{dp^i}{d\tau} = -\Gamma_{00}^i \frac{(p^0)^2}{m} + f^i. \quad (43)$$

As seen “by the particle”, this external force is equivalent to apply a Lorentz force in the opposite direction to the electric field which enters the proper s-t metric, then in general

$$f^i = -q F_0^i \frac{p^0}{m}, \quad \rightarrow \quad \Gamma_{00}^i p^0 = -q F^{i\mu} g_{\mu 0}. \quad (44)$$

Using $p^\mu p_\mu = -m^2 c^2$, we got for the particle at rest

$$m^2 c^2 = -p^0 p_0 = -g_{00} (p^0)^2 \quad \rightarrow \quad p^0 = mc / \sqrt{-g_{00}}, \quad (45)$$

and then,

$$\Gamma_{00}^i = \frac{q}{mc} F^{i\mu} g_{\mu 0} \sqrt{-g_{00}}. \quad (46)$$

3 Applications

In the previous section we have set the basis for obtaining the metric of the proper s-t. In the remaining of this work we will apply such principles to obtain the proper s-t metric defined in two cases of interest: a spherically symmetric electric field and a constant electric field.

3.1 Spherically Symmetric proper s-t

As a first application we will consider a particle of charge q in the presence of an external electric field with spherical symmetry. We assume that exist coordinates t, r, θ, ϕ such that the metric is diagonal

$$ds^2 = -e^a (c^2 dt^2) + e^b dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (47)$$

with a and b functions of r alone.

Since we do not know which are the field equations that would allow us to obtain the metric, we can try to gain some information about $g_{\mu\nu}$ by trying to solve the Maxwell equations

$$\nabla_\mu F^{\nu\mu} = \mu_0 J^\nu = 0, \quad (48)$$

⁹Here, by force we mean any other interaction that is not explicitly include in the geometry of the proper s-t. It can be or not of EM type.

whose only non-trivial solution is

$$\nabla_\mu F^{0\mu} = \partial_r F^{0r} + \Gamma_{\mu r}^\mu F^{0r} = 0. \quad (49)$$

From the Christoffel symbols definition

$$\Gamma_{\rho\sigma}^\mu = \frac{1}{2} g^{\mu\lambda} (-g_{\rho\sigma,\lambda} + g_{\lambda\rho,\sigma} + g_{\sigma\lambda,\rho}), \quad (50)$$

we obtain the following relation valid for a diagonal metric

$$\Gamma_{\mu r}^\mu = \frac{1}{2} g^{\mu\mu} (g_{\mu\mu,r}) = \frac{1}{2} \partial_r \sum_\mu \ln |g_{\mu\mu}| = \frac{1}{2} \partial_r \ln(-g). \quad (51)$$

Therefore, substituting into Eq. (49) yields

$$\partial_r \ln(-gE^2) = 0, \quad g = -e^{(a+b)} r^4 \sin^2 \theta. \quad (52)$$

Since, $E = cF^{0r}$ is the electric field which is independent on the proper s-t as discussed in the previous section, it is totally described by the background s-t and must have the form

$$E = \frac{K}{r^2}, \quad K = \frac{Q}{4\pi\epsilon_0}, \quad (53)$$

for some constant source charge Q . Using this into Eq. (52) we note that $a = -b$ or

$$g^{rr} = -g_{00}, \quad (54)$$

which leads to

$$\Gamma_{00}^r = -\frac{1}{2} g^{rr} g'_{00} = \frac{1}{2} g_{00} g'_{00}. \quad (55)$$

By using the general equation Eq. (46), we arrive at

$$g'_{00} = 2 \frac{qE}{mc^2} \sqrt{-g_{00}}, \quad (56)$$

whose solution is

$$\sqrt{-g_{00}} = 1 + \frac{q}{mc^2} K \left(\frac{1}{r} - \frac{1}{r_0} \right). \quad (57)$$

Here, $r_0 \neq 0$ is some constant which expresses the gauge freedom. In solving this equation the integration constant was chosen in such a way that $g_{00} = -1$ when there's no electric field, $K = 0$. This result was first obtained by Barros [8].

Let's see that this result is in agreement with the linear approximation as it should be. In the linear approximation $g_{00} = \eta_{00} + h_{00}$ with $h_{00} = 2 \frac{q}{m_0 c} A_0$, where A_0 is the electrostatic potential. On the other hand, in the background s-t $\frac{E^r}{c} = F_{r0} = \partial_r A_0$, then

$$A_0 = -\frac{K}{c} \left(\frac{1}{r} - \frac{1}{r_0} \right) \quad \rightarrow \quad h_{00} = -2 \frac{q}{m_0 c^2} K \left(\frac{1}{r} - \frac{1}{r_0} \right), \quad (58)$$

which is exactly the same expression we obtain by expanding g_{00} in Eq. (57).

3.2 Field equations

Now that we do know which is the metric for the spherically symmetric case, we can try to find out a field equation for the proper s-t which allows us to obtain the metric in any case. We propose to search for an Einstein-like equation

$$G_{\mu\nu} = kT_{\mu\nu}, \quad (59)$$

for some constant k , and for a given energy-momentum tensor $T_{\mu\nu}$ which should depend on the interaction of the particle with the EM field.

By using the metric of the previous section, and fixing the gauge, $r_0 = \infty$, we arrive at

$$G^\theta_\theta = G^\phi_\phi = -G^0_0 = -G^r_r = \left(\frac{q}{mc^2}\right)^2 E^2. \quad (60)$$

Note that, in this case the Einstein tensor is traceless, which is also the case for the EM energy-momentum tensor

$$T^\mu_\nu = \frac{1}{\mu_0} \left[-F^\mu_\lambda F^\lambda_\nu + \frac{1}{4} \delta^\mu_\nu F^\alpha_\beta F^\beta_\alpha \right], \quad (61)$$

so, we can try to use this as the source for our field equations. We need however to stress that this is a gauge dependent statement, as Eq. (60) was obtained in a fixed gauge, and it can be shown that for a general r_0 , the metric in Eq. (57) leads to

$$G^\mu_\mu = \frac{2\alpha(2-\alpha)}{r^2}, \quad \alpha = \frac{q}{mc^2} \frac{K}{r_0}, \quad (62)$$

that is, the trace of the Einstein tensor does not vanishes.

In addition, the EM tensor in Eq. (61) should be treated as the EM tensor in the proper s-t not on the background s-t, then indexes should be raised and lowered with the metric $g_{\mu\nu}$ not $\eta_{\mu\nu}$. Computing the Eq. (61) for our case, we have

$$T^\theta_\theta = T^\phi_\phi = -T^0_0 = -T^r_r = \frac{1}{2} \epsilon_0 E^2, \quad (63)$$

where we used $E = cF^{0r}$ and $\mu_0 \epsilon_0 = 1/c^2$. Then, by comparing Eq. (63) with Eq. (60) we have

$$G_{\mu\nu} = \frac{2}{\epsilon_0} \left(\frac{q}{mc^2}\right)^2 T_{\mu\nu}. \quad (64)$$

This result was first obtained by Barros [8]. Now that we are in possession of the field equations let's obtain the metric for the case of a constant electric field.

3.3 Uniform Electric Field

Let's consider an electric field in the z -axis, which is independent of t, x, y . If we assume that the line element is diagonal in this coordinate system, then

$$ds^2 = -e^a (c^2 dt^2) + e^b (dx^2 + dy^2) + e^d dz^2, \quad (65)$$

where a, b and c are functions of z only. By symmetry reasons we choose $g_{xx} = g_{yy}$.

The Maxwell equations leads to

$$\nabla_\mu F^{0\mu} = \partial_z F^{0z} + \Gamma^\mu_{\mu z} F^{0z} = 0, \quad (66)$$

or

$$\ln(e^a e^{2b} e^d E^2) = \text{constant}. \quad (67)$$

If now we set $E = E_0 = \text{constant}$, then, $a + 2b + d = \text{constant}$, and using the fact that $a = b = d = 0$ for $E_0 = 0$, then¹⁰

$$d = -a - 2b. \quad (68)$$

With this result, we may compute the Einstein tensor and the energy-momentum tensor, whose non-vanishing components are

$$G^0_0 = W (a'' + 2a'b' + a'^2), \quad (69)$$

$$G^z_z = W (a'' + 2a'b' + a'^2 + 2b'' + 3b'^2), \quad (70)$$

$$G^x_x = G^y_y = W (b'' + a'b' + 2b'^2), \quad (71)$$

where the prime means a derivative with respect to z , and $T^0_0 = T^z_z = -T^x_x = -T^y_y$ with

$$T^0_0 = -\frac{1}{2\mu_0} (E_0 e^{-b})^2, \quad W = -\frac{1}{2} e^{a+2b}. \quad (72)$$

The field equations Eq. (64) implies $G^0_0 = G^z_z$, then $3b'^2 + 2b'' = 0$, whose solution is

$$e^b = [1 + 3\lambda(z - z_*)]^{\frac{2}{3}}, \quad (73)$$

where the constants z_* and λ have been written in the appropriated way to express invariance under translations and to get the limit $e^b \rightarrow 1$ when $E_0 \rightarrow 0$ (here, $\lambda \rightarrow 0$).

We can now use Eq. (73) into Eq. (71) to get e^a , but we prefer to use the constrain (46), which leads to

$$\begin{aligned} \Gamma^z_{00} &= -\frac{1}{2} g^{zz} g'_{00} = \frac{1}{2} e^{a+2b} (e^a)' \\ &= \frac{q}{m_0 c} e^{a/2} e^a \frac{E_0}{c}, \end{aligned} \quad (74)$$

and by using Eq. (73), we get

$$e^a = \left(\frac{q}{m_0 c^2} \frac{E_0}{\lambda} \right)^2 e^{-b}. \quad (75)$$

Considering that $e^a = e^b = 1$ when $E_0 \rightarrow 0$, then necessarily $\lambda = \pm qE_0/m_0 c^2$, and using $a + d = -2b$, the final result is

$$e^a = e^d = e^{-b} = [1 + 3\lambda(z - z_*)]^{-\frac{2}{3}}, \quad \lambda = \frac{qE_0}{m_0 c^2}, \quad (76)$$

where we took the positive sign λ as it can be always absorbed into E_0 by properly choosing the orientation of the z -axis as being either parallel or anti-parallel to the electric field.

¹⁰Also, this constant can be absorbed into dt^2 .

4 Analyzing the equations of motions

In this section we examine the equations of motion in the proper s-t and their consequences on the dynamics of the particle. The treatment for the spherical case is very similar to the usual one of the Schwarzschild metric, so we do not detail on this here. The case of proper s-t of a particle inside a constant electric field deserves more attention. From the previous section we have for $\vec{E} = E_0 \hat{z}$ (we take without losing generality $E_0 > 0$)

$$ds^2 = \xi (-c^2 dt^2 + dz^2) + \xi^{-1} (dx^2 + dy^2),$$

$$\xi = [1 + 3\lambda(z - z_*)]^{-2/3}, \quad \lambda = \frac{qE_0}{m_0 c^2}. \quad (77)$$

We see that the metric has a singularity at $3\lambda(z - z_*) = -1$. This appears to be totally opposite to what we expect from translational invariance. However, we will now see that initial conditions do not allow the particle to reach such singular point.

As we will show later (see Eq. (82)), setting $v_z = 0$ leads us to a quadratic equation in ξ which gives two solutions, one with $\xi > 0$ and other with $\xi < 0$. The physical solution is that of positive ξ , then necessarily we have

$$\lambda(z_0 - z_*) > -1, \quad (78)$$

where z_0 is the coordinate such that $v_z = 0$.

As we can see intuitively and is shown in Figure 1, a positive charge ($\lambda > 0$) will move to the right of z_0 due to the action of the electric field $E_0 > 0$, and a negative charge will move to the left. So, for $\lambda > 0$ we have $z \geq z_0$, while for $\lambda < 0$ we have $z \leq z_0$ for all z in the particle's trajectory, then we always have $\lambda(z - z_0) \geq 0$. As a consequence

$$\lambda(z - z_*) = \lambda(z - z_0) + \lambda(z_0 - z_*) \geq \lambda(z_0 - z_*) > -1, \quad (79)$$

and therefore the particle never reaches the singularity and in this way it respect the translational invariance. In fact, if we redefine $z \rightarrow \bar{z} = z - w$, then necessarily $z_0 \rightarrow \bar{z}_0 = z_0 - w$ and $z_* \rightarrow \bar{z}_* = z_* - w$; consequently $\bar{z} - \bar{z}_* = z - z_*$.

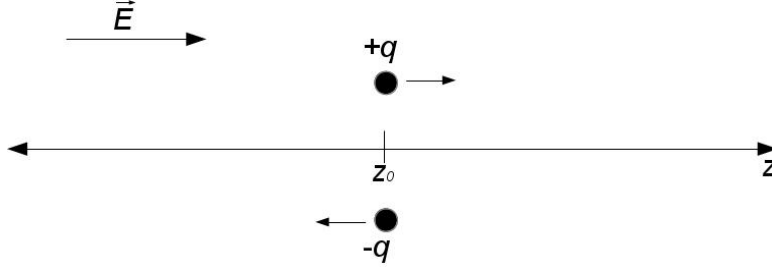


Figure 1: Direction of motion from the return position z_0 .

If for example the particle starts at position z_1 and goes in direction of z_0 , then when arriving at z_0 their velocity will change sign, forcing the particle to go back. We will then call z_0 the return position.

We know (see Carroll [9]) that if the metric does not depend on a given coordinate x^μ , then the corresponding momentum

$$p_\mu = g_{\mu\nu} p^\nu = m g_{\mu\nu} \frac{dx^\nu}{d\tau}, \quad (80)$$

is a constant of motion. So, the conserved quantities associated to the metric Eq. (77) are: $E = -p_0 c/m$, $P = p_x/m$ and p_y .

The normalization condition $g_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) = -1$ is (hereafter we set $p_y = 0$)

$$-1 = \xi \left[-c^2 \left(\frac{dt}{d\tau} \right)^2 + \left(\frac{dz}{d\tau} \right)^2 \right] + \xi^{-1} \left(\frac{dx}{d\tau} \right)^2, \quad (81)$$

and using the relation $E \frac{d}{dt} = c^2 \xi \frac{d}{d\tau}$, $v_z = dz/dt$, we get

$$v_z^2 = c^2 - \frac{P^2}{E^2} c^4 \xi^2 - \frac{\xi}{E^2} c^6. \quad (82)$$

On the other hand $P = \text{constant} = P(z = z_*)$, then

$$P = \frac{1}{\xi} \frac{dx}{d\tau} = \left(\frac{E}{\xi^2 c^2} \frac{dx}{dt} \right) \Big|_{z_*} = \frac{E}{c^2} v_{x*} \quad (83)$$

where we define v_{x*} and v_{z*} as the x and z velocities at $z = z_*$. Now substituting into Eq. (82) yields

$$v_{z*}^2 = c^2 - (v_{x*})^2 - \frac{c^6}{E^2}, \quad \rightarrow \quad E = \frac{c^2}{\sqrt{1 - v_{*}^2/c^2}}, \quad (84)$$

with

$$v_*^2 = v_{z*}^2 + v_{x*}^2. \quad (85)$$

We now concentrate on the case $P = 0$, so $v_z = v$ and from Eq. (82)

$$\xi = \left(1 - \frac{v^2}{c^2} \right) \frac{1}{1 - v_*^2/c^2}, \quad \rightarrow \quad 1 + 3\lambda(z - z_*) = \left(\frac{1 - \frac{v^2}{c^2}}{1 - v_*^2/c^2} \right)^{-3/2}, \quad (86)$$

from which the limit of small velocities $v \ll c$ leads to

$$\lambda(z - z_*) = \frac{v^2}{2c^2} - \frac{v_*^2}{2c^2} + \mathcal{O}(c^{-4}), \quad \rightarrow \quad \frac{1}{2} m_0 (v^2 - v_*^2) = qE_0(z - z_*) + \mathcal{O}(c^{-2}), \quad (87)$$

which is just the energy conservation in the Newtonian limit ($\Delta E_{kinetic} = -\Delta E_{potential}$).

4.1 Comparing with Special Relativity

In the SR, a charged particle in presence of an electric field E_0 satisfies the energy conservation (see [18, 19])

$$qE_0(z - z_*) = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} \Big|_{v_*}^v, \quad (88)$$

which we write as

$$\lambda(z - z_*) = \frac{1}{\sqrt{1 - v^2/c^2}} - \frac{1}{\sqrt{1 - v_*^2/c^2}}. \quad (89)$$

To compare the two approaches, let's take, without losing generality, $z_* = z_0 = 0$, so $v_* = 0$ and then from equations (86) and (89) follow

$$\text{proper s-t: } \rightarrow \quad (1 + 3\lambda z)^{1/3} = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad (90)$$

$$\text{SR: } \rightarrow \quad (1 + \lambda z) = \frac{1}{\sqrt{1 - v^2/c^2}}. \quad (91)$$

To have an idea of which order of magnitude in usual units we are talking about, consider the example given in Bouda and Belabbas [16]. For a dust particle of mass $m \approx 10^{-14} \text{ kg}$, saturation charge¹¹ $q \approx 10^{-18} \text{ C}$, and $c^2 = 9 \times 10^{16} \text{ m}^2/\text{s}^2$, we have

$$\lambda \approx 10^{-21} E_0/V, \quad (92)$$

where V means volts. An electric field as intense as $E_0 \approx 10^8 \text{ V/m}$ is rare in practice (Bouda and Belabbas [16], see also [17]), for such an intense field we have $\lambda \approx 10^{-13}/m$ (here m stands for meter). Then for such a particle moving in a region $|\Delta z|$ as big as a billion meter size the linear approximation works quite well and so Eq. (90) coincides with Eq. (91).

Figure 2 compares the two approaches, continuous lines are the SR solutions while dashed lines are the solutions for the proper s-t. For small values of λz , say $\lambda z < 0.1$ the two cases agree very well. We also see that in the proper s-t formalism, the particle approach the limit velocity $v = c$, slower than in the SR case.

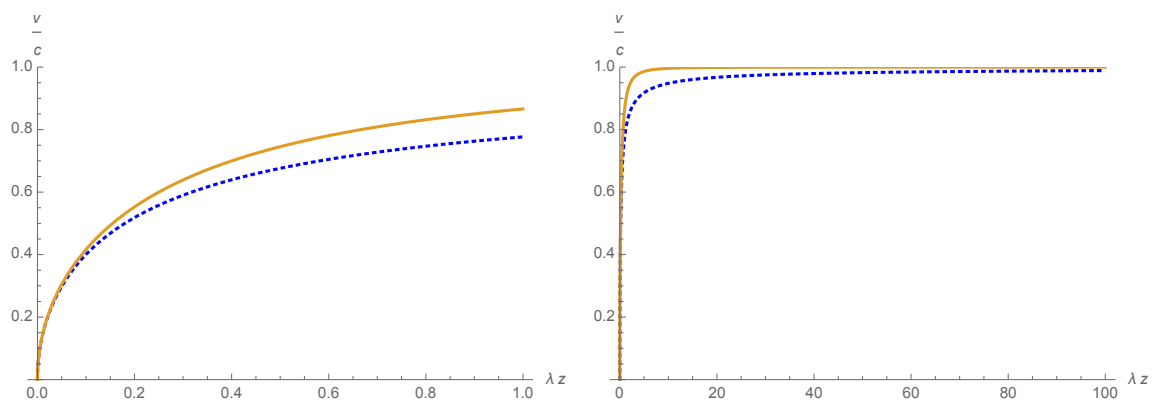


Figure 2: Velocity vs position. Dotted lines are the prediction of Eq. (90) while continuous lines are predicted by SR Eq. (91).

4.2 Solution to the equation of motion

In this section, in order to complete our analysis, we obtain the time dependence of the particle's path. For simplicity we restrict our calculations to the one dimensional case $P = 0$, and again we take $z_* = z_0 = 0$ (or $v_* = 0$).

In this case we have (Eq. (90))

$$u^{-2/3} = 1 - v^2/c^2, \quad \rightarrow \quad cdt = \pm \frac{dz}{\sqrt{1 - u^{-2/3}}}, \quad (93)$$

with $u = 1 + 3\lambda z$. Then, in terms of u , Eq. (93) yields

$$3\lambda cdt = \pm \frac{u^{1/3} du}{u^{2/3} - 1}, \quad \rightarrow \quad 3\lambda c(t - t_0) = \pm \left(2 + u^{2/3}\right) \sqrt{u^{2/3} - 1} \Big|_{u_0}^u \quad (94)$$

¹¹Note that we are not in the quantum limit.

with u_0 the value of u at $t = t_0$. Taking $t_0 = 0$, and using again the variable z we get

$$(3\lambda)(ct) = \pm (f(z) - f(z_0)), \quad (95)$$

$$f(z) = \left[2 + (1 + 3\lambda z)^{2/3} \right] \sqrt{(1 + 3\lambda z)^{2/3} - 1}. \quad (96)$$

This equation gives the time as a function of the position. However we can isolate the variable z as a function of time, and the reader can check that this leads to

$$1 + 3\lambda z = \left[-1 + \frac{2^{2/3}}{2 + s^2 + s\sqrt{4 + s^2}} + \frac{2 + s^2 + s\sqrt{4 + s^2}}{2^{2/3}} \right]^{3/2}, \quad (97)$$

with

$$s = \pm(3\lambda)(ct) + f(z_0). \quad (98)$$

Figure 3 shows a set of geodesics for different initial conditions: $3\lambda z = 0, 1.5, 3$. The cases $3\lambda z = 1.5$ and $3\lambda z = 3$, start with negative velocity and approach to the return point (in this case, the origin), then go back to the $z > 0$ region. The other case is for initial conditions $3\lambda z = 0$ and $v = 0$, so they are always moving away from the origin.

The initial velocities for each case, can be obtained from Eq. (90)

$$\frac{v}{c} = \pm \sqrt{1 - (1 + 3\lambda z)^{-2/3}}, \quad (99)$$

which yields

$$3\lambda z = 1.5 \quad \rightarrow \quad \frac{v}{c} \approx 0.676, \quad (100)$$

$$3\lambda z = 3 \quad \rightarrow \quad \frac{v}{c} \approx 0.776. \quad (101)$$

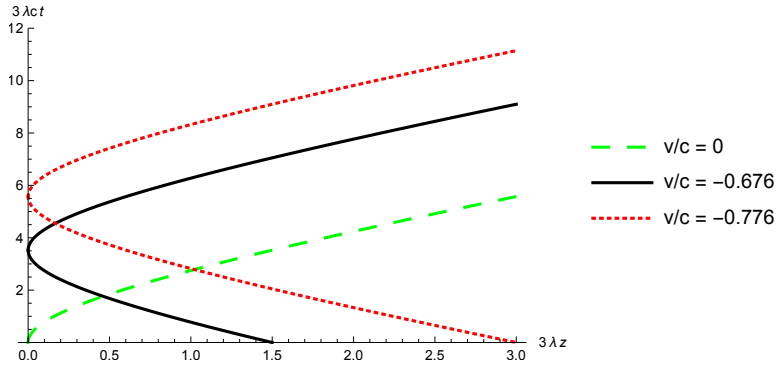


Figure 3: Geodesic curves for the one-dimensional case. Initial conditions shown on the right.

In this plot we explicitly see that due to the initial conditions and the return position, the particle never approach the region $\lambda z < 0$ (here $z_* = 0$), this makes the proper s-t described by Eq. (77) free of singularities. We see also that basically each curve is equal to the other up to a displacement, this agrees with translational invariance as we would expect.

Summary and Conclusions

In this work we have introduced the concept of proper s-t as a geometric tool to describe the interaction between a punctual particle and an electromagnetic external field. In this formalism we have considered the particle path as being a geodesic in a curved space-time, and so, the electromagnetic force could be understood in a purely geometric way.

With those considerations we have seen in section 1.1 that the geodesic equation in the proper s-t leads to the Lorentz force in the limit of weak field and small velocities. This allowed us to relate the metric in the proper s-t with the EM potential. In section 2 we have proposed general statements about the characteristics of the proper s-t beyond the weak field approximation which allowed us to make some applications in section 3. In particular, we have obtained the field equations that define this geometry which are similar to the Einstein's equations, where in general, the energy-momentum tensor has information of both, the particle and the external field. We have seen however, that those field equations violate gauge invariance, as they are only valid in a specific gauge. This is a question that must be studied more carefully and will be left for future works. We have also confirmed that the present treatment is compatible with SR in the limit of small velocities and weak fields (see section 4.1), that means that the general behavior of the particle has been obtained. An interesting result is that it is possible to find connexions to the usual formulation, and the conditions in order to observe these connections are determined. Thinking in these terms, the proposed theory may be seen as a generalization of the usual theory.

Acknowledgments We thank Brazilian research agency CNPq for financial support.

References

- [1] B. P. Abbot et al., Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.* 116, 061102 (2016).
- [2] Barros, C.C.Jr. Quantum mechanics in curved space-time. *Eur. Phys. J. C* 42, 119-126 (2005) [physics/0409064]
- [3] Barros, C.C.Jr. Quark confinement and curved spaces. *Eur. Phys. J. C* 45, 421-425 (2006) [hep-ph/0504179].
- [4] H. Aspden, *Physics Unified*. (Sabberton Publications, 1980).
- [5] L. Barnett, *The Universe and Dr. Einstein Physics Unified*. (William Morrow, New York, 1966).
- [6] W. Pauli, *Theory of Relativity*. (Pergamon Press Ltd, 1958).
- [7] H. Weyl, Electron and Gravitation. 1. (In German), *Z. Phys.* **56**, 330 (1929) [*Surveys High Energ. Phys.* **5**, 261 (1986)].
- [8] Barros, C.C.Jr. Quantum mechanics in curved space-time. II. ArXiv e-print: physics/0509011.
- [9] S.M. Carroll, *Spacetime and geometry: An introduction to general relativity*. (Addison Wesley, 2004).

- [10] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. (John Wiley & Sons, 1972).
- [11] R.M. Wald, General Relativity. (The University of Chicago Press, 1984).
- [12] F. de Felice and C. J. S. Clarke, Relativity on Curved Manifolds. (Cambridge Univ. Press, 1995).
- [13] D.J. Griffiths, Introduction to Electrodynamics. Third Ed. (Pearson Addison Wesley, 1999).
- [14] J.D. Jackson, Classical Electrodynamics. Third Ed. (John Willey & Sons, 1998).
- [15] L. D. Landau & E. M. Lifshitz, The Classical Theory of the Fields, Volume 2. Fourth Ed. (Oxford: Pergamon, 1987).
- [16] A. Bouda and Belabbas, A.: A possible Reinterpretation of Einstein's Equations, Int. J. Theor. Phys. 49, 2630-2639 (2010) [arXiv:1012.2245 [gr-qc]].
- [17] J. Rafelski, L. Labun and Y. Hadad, Horizons of Strong Field Physics, AIP Conf. Proc. 1228, 39 (2010) [arXiv:0911.5556 [physics.gen-ph]].
- [18] J.L. Anderson, Principles of Relativistic Physics. (Academic Press Inc., 1967).
- [19] W. Rindler, Introduction to Special Relativity. (Clarendon Press-Oxford, 1982).